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List monopolar partitions of claw-free graphs

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ABSTRACT

The list monopolar partition problem asks whether a graph G together with lists $L(v) \subseteq \{0, 1\}$, $v \in V(G)$ admits a mapping $f : V(G) \to \{0, 1\}$ such that $f(v) \in L(v)$ for each $v \in V(G)$, $f^{-1}(0)$ induces an independent set and $f^{-1}(1)$ induces a disjoint union of cliques in G. This problem is NP-complete in general. We show that the problem is solvable in time $O(n^2m)$ for claw-free graphs where n and m are the numbers of vertices and edges respectively in the input graph.

1. Introduction

Given a graph G and lists $L(v) \subseteq \{0, 1\}$, $v \in V(G)$, a *list monopolar partition of G with respect to the lists L* is a mapping $f : V(G) \to \{0, 1\}$ such that $f(v) \in L(v)$ for each $v \in V(G)$, $f^{-1}(0)$ induces an independent set and $f^{-1}(1)$ induces a disjoint union of cliques in G.

The concept of list monopolar partitions generalizes monopolar partitions and hence monopolar graphs, which have been studied in [3–9,11]. Indeed, a *monopolar partition* of G is a list monopolar partition with respect to the lists $L(v) = \{0, 1\}$ for all $v \in V(G)$; any graph which has such a partition is called *monopolar*. Monopolar graphs are a common generalization of bipartite and split graphs.

The *list monopolar partition problem* asks whether an input graph G, together with lists $L(v) \subseteq \{0, 1\}$, $v \in V(G)$, admits a list monopolar partition with respect to L. When the input graph G is restricted to have lists $L(v) = \{0, 1\}$ for all $v \in V(G)$, the list monopolar partition problem is called the *monopolar partition problem*.

The monopolar partition problem (and hence the list monopolar partition problem) is NP-complete in general [10]. A graph is *chordal* if it contains no induced cycle of length ≥ 4 and is *claw-free* if it contains no induced $K_{1,3}$. The list monopolar partition problem has been solved in time $O(n + m)$ for chordal graphs [7], and the monopolar partition problem has been solved in time $O(n^4m^2)$ for claw-free graphs [5]. An $O(n^3)$ time algorithm for the latter problem is given in [4].

In this paper, we show that the list monopolar partition problem for claw-free graphs can be solved in time $O(n^2m)$. Our result is in sharp contrast to another result in [4] which shows that it is NP-complete to decide if a claw-free graph G admits a *polar partition*, that is, a mapping $f : V(G) \to \{0, 1\}$ such that $f^{-1}(0)$ induces a complete multipartite graph and $f^{-1}(1)$ induces a disjoint union of cliques. Graphs which admit polar partitions are called *polar graphs* (cf. [2]). Clearly, every monopolar graph is polar.

Given a graph H, the *line graph* $L(H)$ of H is the graph with vertex set $E(H)$ such that two vertices are adjacent in $L(H)$ if and only if the two corresponding edges are adjacent (i.e., have a common endvertex) in H. The graph H is called a *root graph* of $L(H)$.

In [3], the authors gave an $O(n)$ time algorithm for solving the following problem: given a graph H, determine whether the edge set of H can be partitioned into A and B in such a way that A is a matching and B induces a disjoint union of stars

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and triangles in H; a graph which admits such a partition is called *line-monopolar*. The algorithm in [3] also constructs a line-monopolar partition if one exists. Given a line graph with n vertices and m edges, a root graph with $O(n)$ vertices can be found in time $O(n+m)$ as in [12]. Since it is clear that a line graph is monopolar if and only if its root graph is line-monopolar, we have the following theorem.

Theorem 1.1 ([3]). *There is an $O(n+m)$ time algorithm to decide if a line graph is monopolar and to find a monopolar partition if one exists.*

2. Solving the list monopolar partition problem for claw-free graphs

Our strategy for solving the list monopolar partition problem for claw-free graphs is to reduce it to the monopolar partition problem for line graphs. The reduction will consist of three steps which can all be implemented in polynomial time, so by Theorem 1.1 the list monopolar partition problem for claw-free graphs is polynomial time solvable.

A *kite* consists of a 4-cycle $wxyzw$ and the edge xz. We shall call x, z the *center vertices* of the kite. It follows from a characterization of line graphs due to Beineke ([1]; see also [13]) that every claw-free graph which does not contain an induced kite is a line graph.

Lemma 2.1. *Let G be a graph and f be a monopolar partition of G. Then f maps to 1 the center vertices of every induced kite and maps to 0 any vertex which forms an induced P_3 with two center vertices of (possibly different) induced kites.*

Proof. Suppose that w, x, y, z induce a kite where x, z are the center vertices. Since wzy is an induced path in G, f maps at least one of w, y, z to 0. Hence $f(x) = 1$ as x is adjacent to each of w, y, z. Similarly, $f(z) = 1$.

Now suppose that u, v, w form an induced P_3 where v, w are center vertices of some kite(s). From above we know that $f(v) = f(w) = 1$. Hence $f(u) = 0$. □

Lemma 2.2. *Let G^* be the graph obtained from a path $v_1v_2v_3v_4$ by adding vertices v_5, v_6, v_7 and edges $v_5v_1, v_5v_2, v_6v_2, v_6v_3, v_7v_3, v_7v_4$. Then G^* is a line graph that has no monopolar partition.*

Proof. Clearly G^* is claw-free and contains no induced kite, so it is a line graph. Suppose that f is a monopolar partition of G^*. Then either $f(v_2) = 1$ or $f(v_3) = 1$. By symmetry, we may assume $f(v_2) = 1$. Similarly, f must map one of v_1, v_5 and one of v_3, v_6 to 1. Hence f maps the three vertices of an induced path (with v_2 as the middle vertex) to 1, a contradiction to the fact that $f^{-1}(1)$ induces a disjoint union of cliques in G^*. □

The following proposition explains how to reduce the list monopolar partition problem for claw-free graphs to the list monopolar partition problem for line graphs.

Proposition 2.3. *Let G be a claw-free graph with n vertices and m edges with lists $L(v) \subseteq \{0, 1\}$, $v \in V(G)$. Then there is a line graph G' with lists $L'(v) \subseteq \{0, 1\}$, $v \in V(G')$ such that G has a list monopolar partition with respect to L if and only if G' has a list monopolar partition with respect to L'. Moreover, the graph G' contains $O(n)$ vertices and $O(m)$ edges and both G' and L' can be constructed in time $O(n^2m)$.*

Proof. Let T be the set of all center vertices of kites in G; let S be the set containing each vertex which forms an induced P_3 with two vertices in T. If $1 \notin L(x)$ for some $x \in T$, or $0 \notin L(y)$ for some $y \in S$, or any three vertices of T induces a P_3, then let G' be the graph G^* as described in Lemma 2.2 and $L'(v) = \{0, 1\}$ for all $v \in V(G')$. By Lemmas 2.1 and 2.2, neither G has a list monopolar partition with respect to L nor does G' with respect to L'. So assume that $1 \in L(x)$ for all $x \in T$, $0 \in L(y)$ for all $y \in S$, and T induces a disjoint union of cliques in G.

Construct a new graph G' with lists L' from G and L as follows: for each connected component of the subgraph induced by T, delete all but one vertex and set the list of the remaining vertex to be $\{1\}$. For each vertex $y \in S$, let $L'(y) = \{0\}$. Clearly, G' contains no induced kite or claw and hence it is a line graph. Since G' is an induced subgraph of G, it has $O(n)$ vertices and $O(m)$ edges. The graph G' and the lists L' can be constructed in time $O(n^2m)$. Indeed, it takes $O(n^2m)$ to find the set T by checking each edge e of G to see if it is the center edge of any induced kite. In $O(m)$ time, we can find the connected components of the subgraph induced by T and ensure that each is a clique. Finally, finding the set S takes $O(m)$ time: it amounts to checking whether a vertex v has neighbors in two different components, or a neighbor and a nonneighbor in the same component, of the subgraph induced by T.

It remains to show that G has a list monopolar partition with respect to L if and only if G' has a list monopolar partition with respect to L'. Suppose first that f is a list monopolar partition of G with respect to L. Let f' be the restriction of f on $V(G')$. It follows from the definition of L' that $L'(v) = L(v)$ except when $v \in S \cup T$ in which case $L'(v) \subseteq L(v)$. By Lemma 2.1, $f(v) \in L'(v)$ for each $v \in S \cup T$. This together with the fact that G' is an induced subgraph implies that f' is a list monopolar partition of G' with respect to L'.

Suppose that g' is a list monopolar partition of G' with respect to L'. Let g be the extension of g' such that $g(x) = 1$ for each $x \in V(G) \setminus V(G')$. Note that $V(G) \setminus V(G') \subseteq T$. Moreover, each connected component of the subgraph induced by T is a clique and has exactly one vertex in G'. Let y be any such vertex and let x be a vertex of T that is adjacent to y. The definition of L' implies that every vertex $z \in V(G')$ with $g'(z) = 1$ is adjacent to either both x and y or neither. It follows that the subgraph of G induced by the vertices v with $g(v) = 1$ is a disjoint union of cliques. So g is a list monopolar partition of G with respect to L. □

The next proposition explains how to reduce the list monopolar partition problem for line graphs to a list monopolar partition problem for line graphs with the restriction that each list is either $\{0\}$ or $\{0, 1\}$ (i.e. no list is allowed to be the singleton $\{1\}$). The latter problem shall be called the *zero-or-all list monopolar partition problem*.

Proposition 2.4. *Let G be a line graph with n vertices and m edges with lists $L(v) \subseteq \{0, 1\}$, $v \in V(G)$. Then there is a line graph G' and lists L' such that $L'(v) \neq \{1\}$ for all $v \in V(G')$, and G has a list monopolar partition with respect to L if and only if G' has a list monopolar partition with respect to L'. Moreover, the graph G' contains $O(n)$ vertices and $O(m)$ edges and G' and L' can be constructed in time $O(n + m)$.*

Proof. If G contains an induced path xyz such that $L(x) = L(y) = L(z) = \{1\}$, then let $L'(v) = \{0\}$ for every vertex v with $L(v) = \{1\}$ and $L'(v) = L(v)$ otherwise. Then neither with respect to L nor to L', G can have a list monopolar partition. So assume that G contains no such path. Denote by R the subgraph of G induced by the vertices v with $L(v) = \{1\}$. It follows that each connected component of R is a clique. We shall modify each connected component of R and define the associated lists letting, for each vertex $v \in V(G) \setminus V(R)$, $L'(v) = L(v)$. We show that each modification results in a graph G' (sometimes $G' = G$) such that G has a list monopolar partition with respect to L if and only if G' has a list monopolar partition with respect to L'.

Let C be a connected component of R and let H be a root graph of G (i.e., $L(H) = G$). Suppose first that C contains only one vertex x. If x is adjacent to a vertex with list $\{0\}$, then let $L'(x) = \{0, 1\}$. Clearly, G has a list monopolar partition with respect to L if and only if G has a list monopolar partition with respect to L'. So assume that no neighbor of x in G has list equal to $\{0\}$. Let st be the edge of H corresponding to x. The neighborhood of x can be partitioned into S and T such that the vertices of S and T correspond to the edges incident with s and t respectively in H.

Case 1. $|S| = 0$.

Add a new vertex w adjacent to x and let $L'(w) = \{0\}$ and $L'(x) = \{0, 1\}$. The resulting graph G' is a line graph as a root graph of G' can be obtained from H by adding a new vertex adjacent only to s. It is easy to see that every list monopolar partition of G with respect to L can be extended to a list monopolar partition of G' with respect to L' and the restriction of any list monopolar partition of G' with respect to L' to G is a list monopolar partition of G with respect to L. A similar modification can be done when $|T| = 0$.

Case 2. $|S| = |T| = 1$ and the two vertices in $S \cup T$ are adjacent in G.

Add a new vertex w adjacent to x and the vertex in S in G', and let $L'(w) = \{0\}$ and $L'(x) = \{0, 1\}$. A root graph of G' can be obtained from H by adding a new vertex adjacent to s only. Let f be a list monopolar partition of G with respect to L. Then f cannot map both vertices in $S \cup T$ to 0 as they are adjacent in G. A list monopolar partition f' of G' with respect to L' can be defined by letting $f'(w) = 0, f'(S \cup T) = \{1\}$ and $f'(v) = f(v)$ for every other vertex v. On the other hand, the restriction of any list monopolar partition of G' with respect to L' is a list monopolar partition of G with respect to L.

Case 3. $|S| = |T| = 1$ and the two vertices in $S \cup T$ are not adjacent in G.

If any of the two vertices in $S \cup T$ is of degree one in G, then replace the list of that vertex by $\{0\}$ and let $L'(x) = \{0, 1\}$. Clearly, G has a list monopolar partition with respect to L if and only if G has a list monopolar partition with respect to L'. So assume that both vertices in $S \cup T$ are of degree greater than one.

Subcase 3.1. Both vertices in $S \cup T$ are of degree two in G.

Delete x and add three new vertices w, y, z, each with list $\{0, 1\}$, such that y is adjacent to w, z and the vertex in S, and z is adjacent to w and the vertex in T. A root graph of G' can be obtained from H by deleting the edge st and adding two vertices s_1, s_2 and edges s_1s, s_1t, s_1s_2. Let f be a list monopolar partition of G with respect to L. Note that f cannot map both vertices in $S \cup T$ to 1. If f maps both vertices in $S \cup T$ to 0, then f can be extended to a list monopolar partition f' of G' with respect to L' by letting $f'(w) = f'(y) = f'(z) = 1$. If f maps the vertex in S to 0 and the vertex in T to 1, then f can be extended to f' by letting $f'(y) = f'(w) = 1$ and $f'(z) = 0$; similarly, if f maps the vertex in S to 1 and the vertex in T to 0, then f can be extended to f' by letting $f'(y) = 0$ and $f'(w) = f'(z) = 1$. Let f' be any list monopolar partition of G' with respect to L'. Again note that f' cannot map both vertices in $S \cup T$ to 1. If f' maps both vertices in $S \cup T$ to 0, then a list monopolar partition f of G with respect to L can be defined by letting $f(x) = 1$ and restricting f' to $G - x$. If f' maps one of the two vertices in $S \cup T$ to 1, then f can be obtained by letting $f(x) = 1$ and either restricting f' to $G - x$ or letting $f(S \cup T) = \{0\}$ and restricting f' to $G - x - S - T$.

Subcase 3.2. The vertex in S is of degree ≥ 3.

Denote by u the vertex in S. Let $L'(x) = \{0, 1\}, L'(u) = \{0\}$, and $L'(v) = L(v)$ for all other vertices. Since x is not adjacent to any neighbor of u and G is claw-free, the neighbors of u (other than x) induce a clique. Note that any list monopolar partition f of G with respect to L maps at most one neighbor of u to 0. Consequently, $f(u) = 0$ and $f(x) = 1$. Hence G has a list monopolar partition with respect to L if and only if it has one with respect to L'.

Case 4. $|S| = 1$, $|T| \geq 2$ and there is no edge between S and T.

This case can be treated in the same way as for Subcase 3.2 above. That is, let $L'(x) = \{0, 1\}, L'(u) = \{0\}$ for the vertex u in S, and $L'(v) = L(v)$ for all other vertices. Since any list monopolar partition with respect to L maps x to 1 and u to 0, G has a list monopolar partition with respect to L if and only if it has one with respect to L'.

Case 5. $|S| \geq 2$ and there is an edge in G between S and T.

Let $L'(x) = \{0, 1\}$. Every list monopolar partition of G with respect to L' maps x to 1 and hence it is also a list monopolar partition of G with respect to L. A similar modification can be done when $|T| \geq 2$, and there is an edge between S and T in G.

Case 6. $|S| \geq 2$, $|T| \geq 2$ and there is no edge between S and T in G.

Delete x and add two new vertices y, z and the edge yz with $L'(y) = L'(z) = \{0\}$. A root graph of G' can be obtained from H by deleting the edge st and adding three new vertices which induce a path. Note that neither G has a list monopolar partition with respect to L nor G' has a list monopolar partition with respect to L'.

Suppose next that C is a triangle xyz which corresponds to a triangle in H. Since every edge in H is adjacent to either none or exactly two edges in the triangle, every vertex of G is adjacent to either none or exactly two of x, y, z. It follows that each of x, y, z is of degree ≥ 3 in G if and only if it is a center vertex of some induced kite. If x, y, z are each of degree ≥ 3 in G, let $L'(x) = L'(y) = L'(z) = \{0, 1\}$; by Lemma 2.1, G has a list monopolar partition with respect to L if and only if it has a list monopolar partition with respect to L'. If exactly one vertex, say x, is of degree two in G, add a new vertex x' adjacent to x, y and $L'(x) = L'(y) = L'(z) = \{0, 1\}$. A root graph of the resulting graph G' can be obtained from H by adding a new vertex adjacent to the vertex incident with the edges corresponding to x and y in H. Again, x, y, z are all center vertices in G', so G has a list monopolar partition with respect to L if and only if G' has a list monopolar partition with respect to L'. Likewise, if all three vertices x, y, z are of degree two in G, add two new vertices x', y' adjacent to x, z and y, z respectively, and let $L'(x) = L'(y) = L'(z) = \{0, 1\}$. It is easy to see that G' is a line graph, and that it has a list monopolar partition with respect to L' if and only if G has one with respect to L.

Suppose that C corresponds to a star in H consisting of center s and leaves $s_1, s_2, \ldots, s_k$. Let S be the set of vertices in $V(G) \setminus V(C)$ corresponding to the edges in H incident with s. If $S = \emptyset$, then let G' be obtained from G by adding a new vertex w adjacent only to the vertices of C, with lists $L'(w) = \{0\}$ and $L'(v) = L(v)$ for all $v \in V(G)$. Clearly, G has a list monopolar partition with respect to L if and only if G' has a list monopolar partition with respect to L'. So we may assume that $S \neq \emptyset$. We may further assume that $k \geq 2$ as otherwise C contains only one vertex, the case of which has been covered above. Hence the degree of s in H is at least three.

If any two of $s_1, s_2, \ldots, s_k$, say s_i, s_j, are adjacent, then the two vertices of C corresponding to ss_i and ss_j are the center vertices of a kite in G and hence are mapped to 1 under any monopolar partition by Lemma 2.1. In this case, we may modify L by simply changing the lists of the two vertices corresponding to ss_i and ss_j to $\{0, 1\}$. It is easy to see that G has a list monopolar partition with respect to L if and only if G has a list monopolar partition with respect to the modified lists. Hence we may assume that there is no edge between s_i and s_j for any $i \neq j$. Let S_i be the set of vertices in $V(G)$ corresponding to the edges incident with s_i for each $i = 1, 2, \ldots, k$. Delete C from G and add new vertices w, x, y_i with $L'(w) = L'(y_i) = \{0, 1\}$ and $L'(x) = \{0\}$ such that w is adjacent to x and to every vertex in S and y_i is adjacent to x, y_j and to every vertex in S_i for all $1 \leq i \neq j \leq k$. A root graph of G' can be obtained from H by deleting each edge ss_i and adding two vertices t_1, t_2 and edges st_1, t_1t_2, t_2s_i for each $i = 1, 2, \ldots, k$. If f is a list monopolar partition of G with respect to L, then a list monopolar partition f' of G' with respect to L' can be obtained by letting $f'(x) = 0, f'(w) = f'(y_i) = 1$ and by restricting f to $G - C$. If f' is a list monopolar partition of G' with respect to L', then a list monopolar partition f of G with respect to L can be obtained by letting $f(C) = \{1\}$ and restricting f' to $G - C$.

In time $O(n + m)$ we can find the connected components of R and determine whether R contains an induced path with three vertices. A root graph H of G can be computed in time $O(n + m)$ and within the same complexity we can modify each component C and define the associated lists according to the instructions defined above. The resulting graph G' has $O(n)$ vertices and $O(m)$ edges. $\square$

Our final reduction from the zero-or-all list monopolar partition problem to the monopolar partition problem for line graphs is explained in the following proposition.

Proposition 2.5. *Let G be a line graph with n vertices and m edges with lists L such that $L(v) \neq \{1\}$ for all $v \in V(G)$. Then there exists a line graph G' such that G has a list monopolar partition with respect to L if and only if G' has a monopolar partition. Moreover, the graph G' contains $O(n)$ vertices and $O(m)$ edges and can be constructed in time $O(n + m)$.*

Proof. Let R consist of all vertices $v \in V(G)$ with $L(v) = \{0\}$. If any two vertices in R are adjacent in G, then let G' be the graph G^* as described in Lemma 2.2. Then neither G has a list monopolar partition with respect to L nor G' has a monopolar partition. So assume that R is an independent set. Let H be a root graph of G and let z be an arbitrary vertex in R. Then z corresponds to some edge st in H and the neighborhood $N_G(z)$ can be partitioned into S and T such that the vertices of S and T correspond to edges incident with s and t respectively in H. For each such vertex z, we delete z from G and add new vertices u, v, w, x, y such that w, x, y form a triangle, u is adjacent to w, x and every vertex in S, and v is adjacent to w, y and every vertex in T.

A root graph of the resulting graph G' can be obtained from H by deleting edge st and adding a triangle abc such that a is adjacent to s and b is adjacent to t. So G' is a line graph. Since w, x, y are all center vertices of induced kites in G', by Lemma 2.1 any monopolar partition f of G' satisfies $f(w) = f(x) = f(y) = 1$ and hence $f(u) = f(v) = 0$. It follows that G has a list monopolar partition with respect to L if and only if G' has a monopolar partition.

In time $O(n + m)$ we can decide whether R contains a pair of adjacent vertices. A root graph G can be computed in time $O(m)$, and within the same complexity we can construct G' which clearly has $O(n)$ vertices and $O(m)$ edges. $\square$

Combining Propositions 2.3–2.5 and Theorem 1.1, we obtain the following theorem.

Theorem 2.6. *There is an $O(n^2m)$ time algorithm to decide if a claw-free graph with lists admits a list monopolar partition with respect to the lists and to find such a partition if one exists.*

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